Transversal and cotransversal matroids via the Lindström lemma.

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Abstract

It is known that the duals of transversal matroids are precisely the strict gammoids. The purpose of this short note is to show how the Lindström-Gessel-Viennot lemma gives a simple proof of this result.

1

Matroids and duality. A *matroid* $M = (E, \mathcal{B})$ is a finite set E, together with a non-empty collection $\mathcal{B}$ of subsets of E, called the *bases* of M, which satisfy the following axiom: If B_1, B_2 are bases and e is in $B_1 - B_2$, there exists f in $B_2 - B_1$ such that $B_1 - e \cup f$ is a basis.

If $M = (E, \mathcal{B})$ is a matroid, then $\mathcal{B}^* = \{E - B \,|\, B \in \mathcal{B}\}$ is also the collection of bases of a matroid $M^* = (E, \mathcal{B}^*)$, called the *dual* of M.

Representable matroids. Matroids can be thought of as a combinatorial abstraction of linear independence. If V is a set of vectors in $\mathbb{R}^n$ and $\mathcal{B}$ is the collection of maximal linearly independent sets of V, then $M = (V, \mathcal{B})$ is a matroid. Such a matroid is called *representable* over $\mathbb{R}$, and V is called a *representation* of M.

Transversal matroids. Let $A_1, \ldots, A_r$ be subsets of $[n] = \{1, \ldots, n\}$. A *transversal* (also known as *system of distinct representatives*) of $(A_1, \ldots, A_r)$ is a subset $\{e_1, \ldots, e_r\}$ of $[n]$ such that e_i is in A_i for each i. The transversals of $(A_1, \ldots, A_r)$ are the bases of a matroid on $[n]$. Such a matroid is called a *transversal matroid*, and $(A_1, \ldots, A_r)$ is called a *presentation* of the matroid. This presentation can be encoded in the bipartite graph H with "left" vertex set $L = [n]$, "right" vertex set $R = \{\widehat{1}, \ldots, \widehat{r}\}$, and an edge joining j and $\widehat{i}$ whenever j is in A_i. The

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transversals are the r-sets in L which can be matched to R. We will denote this transversal matroid by $M[H]$.

Strict gammoids. Let G be a directed graph with vertex set $[n]$, and let $A = \{v_1, \dots, v_r\}$ be a subset of $[n]$. We say that an r-subset B of $[n]$ *can be linked to* A if there exist r vertex-disjoint directed paths whose initial vertex is in B and whose final vertex is in A. We will call these r paths a *routing* from B to A. The collection of r-subsets which can be linked to A are the bases of a matroid denoted $L(G, A)$. Such a matroid is called a *strict gammoid*.

We can assume that the vertices in A are sinks of G; *i.e.*, that there are no edges coming out of them. This is because the removal of those edges does not affect the matroid $L(G, A)$.

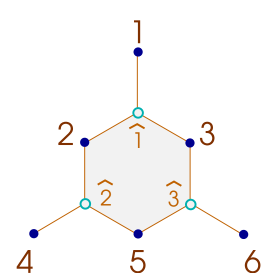
2

Representations of transversal matroids. Consider a collection of algebraically independent α_{ij}s for $1 \leq i \leq r, 1 \leq j \leq n$. Let M be a transversal matroid on the set $[n]$ with presentation $(A_1, \dots, A_r)$. Let X be the $r \times n$ matrix whose (i, j) entry is $-\alpha_{ij}$ if $j \in A_i$ and 0 otherwise. The columns of X are a representation of M.

To see this, consider the columns $j_1, \dots, j_r$. They are independent when their determinant is non-zero. As soon as one of the $r!$ summands in the determinant is non-zero, the determinant itself will be non-zero, by the algebraic independence of the α_{ij}s. But the summand $\pm X_{\sigma_1 j_1} \cdots X_{\sigma_r j_r}$ (where σ is a permutation of $[r]$) is non-zero if and only if $j_1 \in A_{\sigma_1}, \dots, j_r \in A_{\sigma_r}$. So the determinant is non-zero if and only if $\{j_1, \dots, j_r\}$ is a transversal. The desired result follows.

We will find it convenient to choose a transversal $j_1 \in A_1, \dots, j_r \in A_r$ ahead of time, and normalize the rows to have $-\alpha_{ij_i} = 1$ for $1 \leq i \leq r$.

Example 1. Let $n = 6$ and $A_1 = \{1, 2, 3\}, A_2 = \{2, 4, 5\}, A_3 = \{3, 5, 6\}$. The corresponding bipartite graph H is shown below.



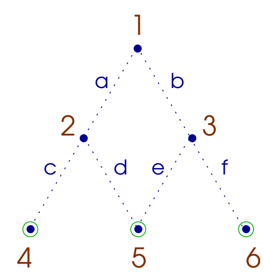
If we choose the transversal $1 \in A_1, 2 \in A_2, 3 \in A_3$, we obtain a representation for the transversal matroid $M[H]$, given by the columns of the following matrix:

$$X = \begin{pmatrix} 1 & -a & -b & 0 & 0 & 0 \\ 0 & 1 & 0 & -c & -d & 0 \\ 0 & 0 & 1 & 0 & -e & -f \end{pmatrix}$$

Representations of strict gammoids. Let $M = L(G, A)$ be a strict gammoid. Say G has vertex set $\{1, \ldots, n\}$ and $A = \{a_1, \ldots, a_{n-r}\}$. Assign algebraically independent weights smaller than 1 to the edges of G_n. For $1 \leq i \leq n - r$ and $1 \leq j \leq n$, let p_{ij} be the sum of the weights of all finite paths[1] from vertex i to vertex j. Let Y be the $(n - r) \times n$ matrix whose (i, j) entry is p_{ji}. The columns of Y are a representation of M.

This is a direct consequence of the Lindström lemma or Gessel-Viennot method, which tells us that the determinant of the matrix with columns $j_1, \ldots, j_{n-r}$ is equal to the signed sum[2] of the routings from $\{j_1, \ldots, j_{n-r}\}$ to $\{a_1, \ldots, a_{n-r}\}$. This signed sum is non-zero if and only if it is non-empty.

Example 2. Consider the graph G shown below, where all edges point down, and the set of sinks $A = \{4, 5, 6\}$.



The representation we obtain for the strict gammoid $L(G, A)$ is given by the columns of the following matrix:

$$Y = \begin{pmatrix} ac & c & 0 & 1 & 0 & 0 \\ ad + be & d & e & 0 & 1 & 0 \\ bf & 0 & f & 0 & 0 & 1 \end{pmatrix}$$

Notice that the rowspaces of X and Y are orthogonally complementary in $\mathbb{R}^6$. That is, essentially, the punchline of this story.

[1]The weight of a path is defined to be the product of the weights of its edges. The sum converges since the weights are less than 1.

[2]The sign is determined by the permutation that matches the starting and ending points of the paths in the routing.

3

Representations of dual matroids. If a rank r matroid M is represented by the columns of an $r \times n$ matrix A, we can think of M as being represented by the r-dimensional subspace $V = \text{rowspace}(A)$ in $\mathbb{R}^n$. The reason is that, if we consider any other $r \times n$ matrix A' with $V = \text{rowspace}(A')$, the columns of A' also represent M.

This point of view is very amenable to matroid duality. If M is represented by the r-dimensional subspace V of $\mathbb{R}^n$, then the dual matroid M^* is represented by the $(n-r)$-dimensional orthogonal complement V^* of $\mathbb{R}^n$.

Digraphs with sinks and bipartite graphs with complete matchings. From a directed graph G on the set $[n]$ and a set of $n-r$ sinks $A \subseteq [n]$ of G, we can construct a bipartite graph H as follows. The left vertex set is $[n]$, and the right vertex set is a copy $[\widehat{n}] - \widehat{A}$ of $[n] - A$. We join $\widehat{u}$ and u for each $u \in [n] - A$, and we join $\widehat{u}$ and v whenever $u \to v$ is an edge of G. This graph H has the obvious complete matching between $\widehat{u}$ and u. Conversely, if we are given the bipartite graph H with a complete matching, it is clear how to recover G and A.

Observe that if we start with the directed graph G and sinks A of Example 1, we obtain the bipartite graph H of Example 2.

Duality of transversal matroids and strict gammoids. Now we show that, in the above correspondence between a graph G with sinks A and a bipartite graph H with a complete matching, the strict gammoid $L(G, A)$ is dual to the transversal matroid $M[H]$. We have constructed a subspace of $\mathbb{R}^n$ representing each one of them, and now we will see that they are orthogonally complementary, as observed in Examples 1 and 2.

Our representation of $M[H]$ is given by the columns of the $r \times n$ matrix X whose (i,i) entry is 1, and whose (i,j) entry is $-\alpha_{ij}$ if $i \to j$ is an edge of G and 0 otherwise. Think of the α_{ij}s as weights on the edges of G. A vector $y \in \mathbb{C}^n$ is in the $(n-r)$-dimensional null space of X when, for each vertex i of G,

$$y_i = \sum_{j \in N(i)} \alpha_{ij} y_j. \tag{1}$$

Here $N(i)$ denotes the set of vertices j such that $i \to j$ is an edge of G.

As before, let p_{ia} be the sum of the weights of the finite paths from i to a in G. Our representation Y of $L(G, A)$ has rows $(y_1, \ldots, y_n) = (p_{1a}, \ldots, p_{na})$ (for $a \in A$). Clearly, each row of Y is a solution to (1), so $\text{rowspace}(Y) \subseteq \text{nullspace}(X)$. But these two subspaces are $(n-r)$-dimensional, so they must be equal, as we wished to show. This completes our proof of the theorem that the strict gammoids are precisely the cotransversal matroids.

4

For more information on matroid theory, Oxley's book [8] is a wonderful place to start. The representation of transversal matroids shown here is due to Mirsky and Perfect [7]. The representation of strict gammoids that we use was constructed by Mason [6] and further explained by Lindström [5][3]. The theorem that strict gammoids are precisely the cotransversal matroids is due to Ingleton and Piff [3]. Our proof of this result appears to be new.

This note is a small side project of [1]. While studying the geometry of flag arrangements and its implications on the Schubert calculus, we were led to study a specific family of strict gammoids which starts with Example 2. I would like to thank Sara Billey for several helpful discussions, and Laci Lovasz and Jim Oxley for help with the references.

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[3] It is in this context that he discovered what is now known as the Lindström lemma or Gessel-Viennot method [2]. This method was also used earlier by Karlin and MacGregor [4].